2023

MATHEMATICS — HONOURS

Paper : CC-13

(Metric Space and Complex Analysis)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$ denote the set of all natural, real, complex and rational numbers respectively.
(Notations and symbols have their usual meanings.)

1. Answer all the following multiple choice questions. For each question **1** mark for choosing correct option and **1** mark for justification : 2×10

(a) Which one of the following is not a metric on $C[0, 1]$, where $C[0, 1]$ is the collection of all real valued continuous functions defined on [0, 1]?

(i) $d(x,y) = \sup\limits_{0\le t\le 1} |x(t)-y(t)|$ (ii) $d(x,y) = \inf\limits_{0\le t\le 1} |x(t)-y(t)|$

(iii) $d(x,y) = \int_0^1 |x(t)-y(t)|\,dt$ (iv) $d(x,y) = \left\{\int_0^1 \left(x(t)-y(t)\right)^2 dt\right\}^{\frac{1}{2}}$.

(b) Let $Y = [1, 2] \cup (3, 4)$. We consider Y as metric subspace of the real line. Then

(i) $[1, 2]$ is closed in Y but not open in Y

(ii) $(3, 4)$ is open in Y but not closed in Y

(iii) $[1, 2]$ is closed in Y as well as open in Y

(iv) None of these.

(c) Let X be an infinite set and $d : X \times X \to \mathbb{N} \cup \{0\}$ be a metric on X. Then every singleton set in (X, d) is

(i) open but not necessarily closed (ii) closed but not necessarily open

(iii) both open and closed (iv) neither open nor closed.

(d) Choose the set Y which as a subspace of $\mathbb{R}^2$, with usual metric, is not complete.

(i) $Y = \{(x, y) \in \mathbb{R}^2 : y = x\}$ (ii) $Y = \mathbb{N} \times \mathbb{N}$

(iii) $Y = \{(x, y) \in \mathbb{R}^2 : |x| = 1\}$ (iv) $Y = \left\{\left(\frac{1}{m}, \frac{1}{n}\right) \in \mathbb{R}^2 : m, n \in \mathbb{N}\right\}$.

(e) The set $\left\{\dfrac{x^2}{1+x^2} : x \in \mathbb{R}\right\}$ is

(i) connected but not compact in $(\mathbb{R}, d_u)$

(ii) compact but not connected in $(\mathbb{R}, d_u)$

(iii) compact and connected in $(\mathbb{R}, d_u)$

(iv) Neither compact nor connected in $(\mathbb{R}, d_u)$.

[Here d_u denotes the usual metric on $\mathbb{R}$].

(f) Under the transformation $w = \dfrac{1}{z}$, the image of the region $\{z = x + iy : x > 1\}$ is transformed into

(i) a circle (ii) a half plane

(iii) interior of a circle (iv) exterior of a circle.

(g) Let $f(z) = |z|^2 z$, $z \in \mathbb{C}$. Which of the following is true?

(i) f is nowhere differentiable in $\mathbb{C}$

(ii) f is differentiable everywhere in $\mathbb{C}$

(iii) f is differentiable everywhere in $\mathbb{C}$ except $z = 0$

(iv) f is differentiable only at $z = 0$ in $\mathbb{C}$.

(h) The radius of convergence of the power series $\sum \dfrac{z^{4n}}{4n+1}$ is

(i) 4 (ii) 1

(iii) ½ (iv) ¼.

(i) What is the value of $\displaystyle\int_{|z|=1} \frac{e^z}{z^2 - 5z + 6}\, dz$?

(i) 0 (ii) $2\pi e^3 i$

(iii) $\pi i e^3$ (iv) $2\pi i$.

(j) What is the maximum possible number of fixed points of a non-identity Mobius transformation in $\mathbb{C}_\infty$?

(i) 0 (ii) 1

(iii) 2 (iv) infinite.

Unit - 1

(Metric Space)

Answer ***any five*** questions.

2. Let (X, d) be a metric space and let $A, B \subseteq X$. Then show that

(a) diam $(A \cup B) \leq \text{diam}(A) + \text{diam}(B) + d(A, B)$

(b) $(A \cap B)^{o} = A^{o} \cap B^{o}$. 3+2

3. Let (Y, d_Y) be a metric subspace of a metric space (X, d). Let $A \subseteq Y$. Prove that interior of A in (X, d) is a subset of interior of A in (Y, d_Y). Give example to show that the equality may not hold. 3+2

4. Show that a sequence $\{x_n\}$ in $(C[0,1], d)$, where $C[0, 1]$ has the usual meaning and $d(x,y) = \sup\limits_{0 \leq t \leq 1} |x(t) - y(t)|$, $\forall x, y \in C[0, 1]$, converges to a function $z \in C[0, 1]$ if and only if the sequence $\{x_n\}$ converges uniformly to z on $[0, 1]$. 5

5. Let (X, d) be a complete metric space and $\{F_n\}$ be a sequence of non-empty closed sets such that $F_{n+1} \subseteq F_n$ for all n. If $\text{diam}(F_n) \to 0$ as $n \to \infty$, then prove that $\bigcap\limits_{n=1}^{\infty} F_n$ contains exactly one element. Is the statement valid for $(\mathbb{Q}, d_u)$? (d_u denotes the usual metric). 4+1

6. (a) Let (X, d_X), (Y, d_Y) be two metric spaces and $A \subseteq X$. For a function $f: A \to Y$ and $a \in A$, it is given that whenever a sequence $\{x_n\}$ in A converges to 'a', the sequence $\{f(x_n)\}$ converges to $f(a)$. Prove that f is continuous at 'a'.

(b) Let (X, d) be a metric space and $A \subseteq X$. Define $f: X \to \mathbb{R}$ by $f(x) = d(x, A)$. Prove that f is uniformly continuous on X. 3+2

7. Let (X, d) be a metric space. Then prove that the following statements are equivalent.

(a) (X, d) is disconnected.

(b) There exists a continuous mapping of (X, d) onto the discrete two element space (X_0, d_0). 5

8. (a) Prove that a compact subset of a metric space (X, d) is closed and bounded.

(b) Let $\{x_n\}$ and $\{y_n\}$ be two sequences in a metric space (X, d) such that $\{x_n\}$ is Cauchy and $\lim\limits_{n \to \infty} d(x_n, y_n) = 0$. Show that $\{y_n\}$ is also Cauchy. 3+2

9. Let (X, d) be a complete metric space and let f be a contraction mapping on X. Prove that there exists one and only one point x in X such that $f(x) = x$. 5

Unit - 2

(Complex Analysis)

Answer ***any four*** questions.

10. (a) Show that the stereographic projections of the points Z and $\frac{1}{\overline{Z}}$ are reflections of each other in the equatorial plane of the Riemann sphere.

(b) Show that the transformation $w = \frac{1-z}{1+z}$ transforms $|w| \leq 1$ into the right half plane $\text{Re}(z) \geq 0$. 3+2

11. (a) Check whether $\lim_{z \to 0} \frac{\overline{z}}{z}$ exists or not.

(b) If $f(z)$ is an analytic function, then show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$. 2+3

12. Let $f: G \to \mathbb{C}$, where $f(x+iy) = u(x, y) + iv(x, y)$ be a function of a complex variable on a region G. Let $u(x, y)$, $v(x, y)$ be differentiable at (x_0, y_0) and let Cauchy-Riemann equations are satisfied at (x_0, y_0). Prove that f is differentiable at $z = x_0 + iy_0$. 5

13. (a) Prove that $f(z) = e^{\overline{z}}$ is nowhere differentiable.

(b) If $f(z)$ and $\overline{f(z)}$ are analytic in a region D, show that $f(z)$ is constant in D. 2+3

14. (a) Find the bilinear transformation that maps the points $z_1 = -i$, $z_2 = 0$, $z_3 = i$ onto the points $w_1 = -1$, $w_2 = i$, $w_3 = 1$. Into what curve is the imaginary axis $x = 0$ transformed?

(b) Prove that if the origin is a fixed point of a bilinear transformation, then the transformation can be written in the form, $w = \frac{z}{cz+d}(d \neq 0)$. (3+1)+1

15. (a) If a power series $\sum a_n z^n$ converges for $z = z_0 (\neq 0)$, prove that it converges absolutely for all z such that $|z| < |z_0|$.

(b) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \left(\frac{iz-1}{2+i}\right)^n$. 2+3

16. (a) Evaluate $\int_C \frac{z\,dz}{(16-z^2)(z+i)}$, where C is the circle $|z| = 2$ taken in the positive sense.

(b) Find the maximum value of the integral $\left|\int_\gamma \frac{dz}{z^2+4}\right|$, where $\gamma(t) = Re^{it}$ for $0 \leq t \leq \pi$ and $R > 2$.

2+3